A Note on Tectonomagnetic Modeling in a Viscoelastic Half-Space: Application to the Mogi Model

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Abstract

This paper presents some theoretical basis for piezomagnetic modeling in the viscoelastic material. The piezomagnetic field of the Mogi model in a viscoelastic half-space is presented as an example by extending its analytic solution in an elastic medium. The reason why the piezomagnetic parameter, i.e. the stress sensitivity β, is independent of the viscoelasticity is argued. Since the piezomagnetization is simply proportional to the deviatoric stress, the correspondence principle is also applicable to the piezomagnetic solution for the elastic problem. The Laplace transform of the viscoelastic parameter is carefully derived based on Fung's textbook, because there has been some confusion among the literature. The Standard Linear Solid (SLS) including the Maxwell rheology as its special case is assumed only for the shear stress, while the bulk modulus K is kept constant. Two different types of the sources for the Mogi model are considered, i.e. the center of pressure (COP) and the center of dilatation (COD), both of which give the same deformation in the case of the elastic medium. The piezomagnetic potential is subject to the correspondence principle, in which the Laplace transform of the piezomagnetic solution is obtained in the same way as the displacement field in the viscoelastic medium. The time-dependent behavior of the piezomagnetic field is obtained by inverse Laplace transformation.

1 Introduction

Studies on the deformation of a viscoelastic earth were developed since 1970's in order to explain very slow crustal deformations with the duration time of several days to a few years, e.g. PELTIER (1974). The viscoelastic behavior of the earth associated with an earthquake was first investigated theoretically by RUNDLE (1978), in which the earth consists of an elastic lithosphere underlain by a viscoelastic asthenosphere. Since the multi-layer earth model was rather difficult to deal with analytically, a more simple case study was anticipated in order to realize the behavior of a viscoelastic earth. BONAFEDE *et al.* (1986) presented the crustal deformation due to the Mogi model in a viscoelastic half-space.

Currenti (*personal communication*, 2006) attempted to extend tectonomagnetic modeling to viscoelastic materials. She followed BONAFEDE *et al.*'s (1986) scheme and applied it to SASAI's (1991a) solution for the piezomagnetic field due to the Mogi model in the elastic medium. She successfully derived an analytical solution of the piezomagnetic field in a viscoelastic medium. During the course of study, however, there were several issues to be clarified, which were discussed between Currenti and Sasai. First of all, we had to verify our standpoint that the piezomagnetic parameter, i.e. the stress sensitivity, is independent of viscoelastic parameters and hence independent of time. The second is that BONAFEDE *et al.*'s (1986) elastic field for the Mogi model had some defect as was revised by BONAFEDE (1990), which should be taken into account in our modeling. The third is that there was some discrepancy among some papers in the definition of the Laplace transform of viscoelastic parameters, which originated from the vague description of the correspondence principle. Moreover, most of them quoted FUNG's (1965) text book as the reference to the Laplace transform of the rigidity, but no explicit formula for that is given there. We had to follow the derivation of the Laplace-transformed rigidity and present the correspondence principle so as not to conflict with each other.

This article mainly describes how we resolved these issues before solving the problem. Then we will briefly outline the derivation process of the solution. The final result will be published elsewhere. (cf. CURRENTI *et al.*, 2007)

2 The constitutive law for the linear viscoelastic solid

The mechanical properties of the viscoelastic solid were fully investigated by FUNG (1965). Although the viscoelastic material generally shows complicated mechanical behavior owing to its non-linearity and anisotropy, we here deal with the simplest case, i.e. the isotropic linear viscoelasticity. In the following sessions, u_i, e_{ij}, σ_{ij}, X_j and ρ denote the displacement, strain, stress, the Cartesian component of the body force per unit volume and density, respectively. The summation convention applies in the following formulas.

A material whose stress is related to strain via such a convolution integral is called the linear viscoelastic solid as:

$$\sigma_{ij} = \int_{-\infty}^{t} G_{ijkl}(x, t-\tau)\frac{\partial e_{kl}}{\partial \tau}(x, \tau)d\tau\,, \tag{1}$$

where G_{ijkl} is a fourth rank tensor, which is called the tensorial relaxation function. This constitutive law is called the stress-strain relation of the relaxation type. Its inverse relation

$$e_{ij} = \int_{-\infty}^{t} J_{ijkl}(x, t-\tau)\frac{\partial \sigma_{kl}}{\partial \tau}(x, \tau)d\tau \tag{2}$$

is called the stress-strain relation of the creep type. The fourth rank tensor J_{ijkl} is the tensorial creep function.

If G_{ijkl} does not change when we rotate the Cartesian coordinates system, such a material is called isotropic. The isotropic fourth rank tensor can be expressed by

$$G_{ijkl} = \frac{G_2 - G_1}{3}\delta_{ij}\delta_{kl} + \frac{G_1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,, \tag{3}$$

where G_1 and G_2 are scalar functions. Substituting eq. (3) into eq. (1), we obtain

$$\sigma'_{ij} = \int_{-\infty}^{t} G_1(x, t-\tau)\frac{\partial e'_{ij}}{\partial \tau}(x, \tau)d\tau, \quad \sigma_{kk} = \int_{-\infty}^{t} G_2(x, t-\tau)\frac{\partial e_{kk}}{\partial \tau}(x, \tau)d\tau\,. \tag{4}$$

Similarly, the isotropic creep function can be given by

$$J_{ijkl} = \frac{J_2 - J_1}{3}\delta_{ij}\delta_{ij} + \frac{J_1}{2}(\delta_{ik}\delta_{ji} + \delta_{il}\delta_{jk})\,, \tag{5}$$

and eq. (2) is reduced to

$$e'_{ij} = \int_{-\infty}^{t} J_1(x, t-\tau)\frac{\partial \sigma'_{kl}}{\partial \tau}(x, \tau)d\tau, \quad e_{kk} = \int_{-\infty}^{t} J_2(x, t-\tau)\frac{\partial \sigma_{kk}}{\partial \tau}(x, \tau)d\tau\,. \tag{6}$$

σ'_{ij} and e'_{ij} are the deviatoric stress and the deviatoric strain, respectively, which are defined by

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}, \quad e'_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}e_{kk}. \tag{7}$$

There are three kinds of linear viscoelastic models which can be described by eq. (1) and/or (2): (a) Maxwell solid, (b) Voigt solid, and (c) standard linear solid (SLS or Kelvin solid) as shown in Fig. 1. In these models, a spring indicates the instantaneous elastic reaction, while a dashpot the gradual stress relaxation proportional to its deformation velocity. Then the third type of the constitutive law can be introduced as

$$P_1(D)\sigma'_{ij} = Q_1(D)e'_{ij}, \quad P_2(D)\sigma'_{kk} = Q_2(D)e'_{kk}, \tag{8}$$

where $D = \partial/\partial t$, and P_1, Q_1, P_2 and Q_2 are polynomials of the differential operater D. These polynomials are defined depending on the type of viscoelastic models. Eq. (8) is particularly useful to obtain the Laplace transform of material parameters. (In Fung's (1965) textbook, a full chapter (Chapter 13: Irreversible Thermodynamics and Viscoelasticity) is devoted to derive the differential operater law (8). The derivation process is rather sophisticated. We present here only the final result.)

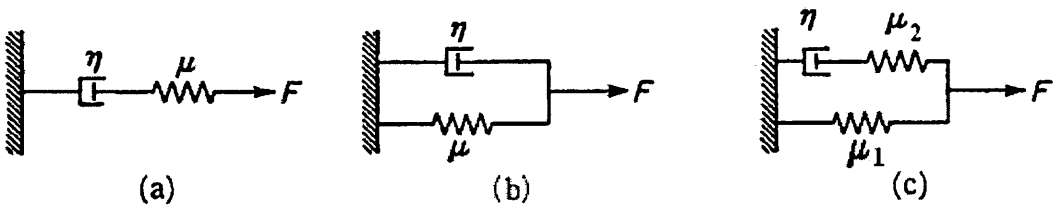

Fig. 1 Three kinds of linear viscoelastic models. (a) Maxwell solid. (b) Voigt solid. (c) Standard linear solid (SLS).

3 Basic equations for the isotropic linear viscoelastic solid

Let us present the governing equations for the displacement field of the isotropic linear viscoelastic solid. The strain is defined as

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{9}$$

The equation of continuity is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho \frac{\partial u_i}{\partial t}) = 0. \tag{10}$$

The equation of motion is given by

$$\sigma_{ij,j} + X_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad \sigma_{ij} = \sigma_{ji}. \tag{11}$$

The stress-strain relationship can be given by either of the relaxation law (4), or the creep law (5) or the differential operator law (8). Since we assume a homogeneous medium, the material functions in (4) and (5) as well as the differential operator with time in (8) are independent of the position. The initial condition is

$$u_i = e_{ij} = \sigma_{ij} = 0 \qquad (\text{for } -\infty < t < 0). \tag{12}$$

The boundary conditions are: (1) to give the traction on the surface S_σ with unit normal ν_j

$$T_i = \sigma_{ij}\nu_j = f_i \quad (\text{on } S_\sigma), \tag{13}$$

or (2) to give the displacement on the surface S_u

$$u_i = g_i \quad (\text{on } S_u), \tag{14}$$

where f_i and g_i are given functions of position and time, while $S_\sigma + S_u = S$ is the whole surface of the material.

The problem of the linear viscoelasticity is thus formulated, in which we solve these equations under the given X_i, f_i, G_i and the initial condition (12). Only except for the constitutive law (4), or (5) or (8), the equations are quite the same as those for the elasticity problem.

4 The correspondence principle

The constitutive relation for the isotropic linear elasticity is Hooke's law:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}, \tag{15}$$

where λ and μ are Lamè's constants. This formula can be rewritten as

$$\sigma'_{ij} = G_1 e'_{ij}, \quad \sigma_{kk} = G_2 e_{kk}, \tag{16}$$

where G_1 and G_2 are the elastic constants corresponding to the viscoelastic parameters $G_1(x, t)$ and $G_2(x, t)$ in eq. (3). They are given by

$$G_1 = 2\mu, \quad G_2 = 3\lambda + 2\mu. \tag{17}$$

Taking the Laplace transform of eq. (4), we have

$$\tilde{\sigma}'_{ij}(s) = s\tilde{G}_1(s)\tilde{e}'_{ij}(s), \quad \tilde{\sigma}_{kk}(s) = s\tilde{G}_2(s)\tilde{e}_{kk}(s), \tag{18}$$

where the tilde indicates the Laplace transform and s the Laplace transform variable. Putting

$$\tilde{\mu}(s) = \frac{1}{2}s\tilde{G}_1(s), \quad \tilde{\lambda}(s) = \frac{1}{3}s\{\tilde{G}_2(s) - \tilde{G}_1(s)\}, \tag{19}$$

we obtain

$$\tilde{\sigma}'_{ij}(s) = 2\tilde{\mu}(s)\tilde{e}'_{ij}(s), \quad \tilde{\sigma}_{kk}(s) = \{3\tilde{\lambda}(s) + 2\tilde{\mu}(s)\}\tilde{e}_{kk}(s). \tag{20}$$

Both the relations are equivalent to

$$\tilde{\sigma}_{ij}(s) = \tilde{\lambda}(s)\tilde{e}_{kk}(s)\delta_{ij} + 2\tilde{\mu}(s)\tilde{e}_{ij}(s). \tag{21}$$

Taking Laplace transforms of all the basic equations for the viscoelasticity (9) to (14), we obtain quite the same formulas as those of linear elasticity. Thus we have equations for viscoelastic materials in the Laplace domain, which are formally quite identical to those for the elastic medium. Then the following *correspondence principle* was proposed (LEE, 1955): *Replace the elastic constants λ and μ in the elastic solution $u_e(x,\, t)$ by their transformed quantities $\tilde{\lambda}(s)$, $\tilde{\mu}(s)$ obtaining $\tilde{u}_e(x,\, s)$, then invert. The function $u(x,\, t)$ is the solution to the viscoelastic problem.*

Let us consider a standard linear solid model as shown in Fig. 1 (c). σ and e represent the load and the strain, respectively. For a spring the load is proportional to the strain, i.e. $\sigma' = 2\mu e'$, while for a dashpot it is proportional to the strain rate, i.e. $\sigma' = 2\eta De'$ ($D = \partial/\partial t$) and η is the viscosity. The factor 2 is multiplied in the case of the deviatoric stress-strain relationship corresponding to eq. (17). (In the following derivation process, the factor 2 in the relationship $2\mu e' = \sigma'$ is crucial. Otherwise, a factor 1/2 remains in the final expression for $\tilde{\mu}(s)$ (see eq. (26)), which is not consistent with the initial value of the Laplace-inverted solution.) Since

$$2\mu_1 e_1' = \sigma', \quad 2\mu_2 e_2' + 2\eta De_2' = \sigma', \quad e' = e_1' + e_2',$$

we have

$$\left(\frac{1}{2\mu_1} + \frac{1}{2\mu_2 + 2\eta D}\right)\sigma' = e'.$$

Applying this relation to the differential operater law eq. (8), we obtain

$$P_1(D) = \eta D + (\mu_1 + \mu_2), \quad Q_1(D) = 2\mu_1(\eta D + \mu_2). \tag{22}$$

As the initial condition (12), we assume a simple case of the Heaviside step function:

$$g(t) = \begin{cases} 1 & (\, t > 0) \\ 0 & (\, t < 0) \end{cases} . \tag{23}$$

Then the Laplace transform of eq. (8) can be obtained simply by replacing D with s as

$$P_1(s)\tilde{\sigma}_{ij}'(s) = Q_1(s)\tilde{e}_{ij}'(s), \tag{24}$$

which corresponds to the first equation of eq. (18). We obtain the Laplace transform of G_1 for SLS as

$$\tilde{G}_1(s) = \frac{Q_1(s)}{sP_1(s)} = \frac{2}{s}\frac{\mu_1(s + \mu_2/\eta)}{s + (\mu_1 + \mu_2)/\eta}. \tag{25}$$

Finally, the Laplace transform of μ is given by

$$\tilde{\mu}(s) = \frac{1}{2}s\tilde{G}_1 = \frac{\mu_1(s + \mu_2/\eta)}{s + (\mu_1 + \mu_2)/\eta}. \tag{26}$$

In some papers, eq. (26) is frequently referred as due to FUNG (1965). However, this formula is not explicitly given in his textbook.

In the case of the Maxwell solid, we obtain by putting $\mu_2 = 0$ in eq. (26):

$$\tilde{\mu}(s) = \frac{\mu_1 s}{s + \mu_1/\eta}. \tag{27}$$

PELTIER (1974) derived $\tilde{\mu}(s)$ for a Maxwell solid in a different way from FUNG (1965), which was quite identical to eq. (27).

The second formula in eq. (16) gives the definition of the bulk modulus K:

$$K = \frac{1}{3}G_2 = \lambda + \frac{2}{3}\mu. \tag{28}$$

In the actual Earth, ordinary rocks behave elastically under the hydrostatic pressure. Hence we follow BONAFEDE *et al.* (1986), in which they remain the bulk modulus K as a constant in the viscoelastic material. The Laplace transform of λ can be obtained as

$$\tilde{\lambda}(s) = K - \frac{2}{3}\tilde{\mu}(s) = K - \frac{2}{3}\frac{\mu_1(s + \mu_2/\eta)}{s + (\mu_1 + \mu_2)/\eta}. \tag{29}$$

However, some authors adopted different rheological models: For example, Peltier (1974) assumed SLS for λ and Maxwell solid for μ, while RUNDLE (1978) a constant value for λ and Maxwell solid for μ.

5 Piezomagnetic effect in the viscoelastic medium

No piezomagnetic experiments have ever been made on the viscoelastic behavior of magnetized rocks. Piezomagnetic properties are carried by titano-magnetite, which occupies only a small portion of ordinary rocks. We suppose that titanomagnetite itself is elastic, while the non-magnetic surrounding matrix behaves viscoelastically. And piezomagnetization is determined simply by the stress field around titanomagnetites. Then the stress sensitivity β is independent of elastic constants and the rheological behavior of the host rock.

The linear piezomagnetic law or the stress-magnetization relationship is given by (SASAI, 1980),

$$\Delta J_i = \frac{3}{2}\beta\sigma'_{ij}J_j, \tag{30}$$

where $\mathbf{J}$ and $\Delta\mathbf{J}$ are the magnetization of the material and its increment (i.e. piezomagnetization), while β is the stress sensitivity. The magnetic potential produced by the piezomagnetization (30) is given by

$$W_i(\mathbf{r}) = \iiint_V \Delta\mathbf{J}_i \cdot \nabla\left(\frac{1}{\rho}\right)dV(\mathbf{r}'), \tag{31}$$

where $\rho = |\mathbf{r} - \mathbf{r}'|$, and W_i indicates the piezomagnetic potential produced by the i-th component of the magnetization.

According to our assumption, β and $\mathbf{J}$ are independent of the rheological behavior of host rocks. When the magnetic parameters β and $\mathbf{J}$ are homogeneous within the volume V, they are excluded from the integral sign. Then only σ'_{ij} determines the magnetic potential, in which the deviatoric stress σ'_{ij} is given as has been described in the previous section. This implies that: *If we already have an analytic solution for the piezomagnetic potential of any linear elastic model, we can also apply the correspondence principle to the solution to obtain the viscoelastic behavior of the piezomagnetic field.*

Since we assume the Heaviside step function (23) as the source time function, we may put

$$\tilde{w}_i(\mathbf{r}, s) = \tilde{g}(s)\tilde{W}_i(\mathbf{r}, s) = \frac{1}{s}\tilde{W}_i(\mathbf{r}, s), \tag{32}$$

where $\tilde{w}_i(\mathbf{r}, s)$ is the Laplace-transformed piezomagnetic potential, while $\tilde{W}_i(\mathbf{r}, s)$ indicates the function $W_i(\mathbf{r})$ in which λ and μ are replaced with $\tilde{\lambda}(s)$ and $\tilde{\mu}(s)$.

6 Displacement field of the Mogi model

BONAFEDE *et al.* (1986) reexamined MARUYAMA's (1964) formulas for the elastic dislocations to obtain those expressed with Lame's constants λ and μ explicitly and finally derived the viscoelastic solution for the Mogi model. However, BONAFEDE (1990) pointed out that the displacement field of the Mogi model as derived from MARUYAMA's (1964) Galerkin vector by himself (BONAFEDE *et al.*, 1986) should be revised to the one based on Love's strain function (MINDLIN and CHENG, 1950), both of which are different by a factor of elastic constants. The solution based on the Galerkin vector gives the deformation of an elastic half-space due to inflation of a spherical shell which involves an elastic sphere inside (BONAFEDE, 1990). This is reasonable because MARUYAMA's (1964) Galerkin vector defines the strain nuclei along the dislocation surface: If the dislocation surface makes a closed one, an elastic medium should exist within the closed surface. A portion of the incremental internal pressure is used so as to compress the sphere, which reduces the surface deformation by a factor of 1/1.8 (in case of $\lambda = \mu$), as compared with the solution based on Love's strain function (BONAFEDE, 1990).

SASAI's (1979, 1991a) piezomagnetic solution was derived from Love's strain function, which is quite coincident with BONAFEDE's (1990) revision. Hence we begin with the following displacement field of the Mogi model (SASAI, 1991a):

$$u_x = \frac{C}{2\mu}\left\{\frac{x}{R_1^3} + \frac{\lambda+3\mu}{\lambda+\mu}\frac{x}{R_2^3} - \frac{6xz(z+D)}{R_2^5}\right\}, \tag{33}$$

$$u_y = \frac{C}{2\mu}\left\{\frac{y}{R_1^3} + \frac{\lambda+3\mu}{\lambda+\mu}\frac{y}{R_2^3} - \frac{6yz(z+D)}{R_2^5}\right\}, \tag{34}$$

$$u_z = \frac{C}{2\mu}\left\{\frac{z-D}{R_1^3} + \frac{(\lambda-\mu)z-(\lambda+3\mu)D}{(\lambda+\mu)R_2^3} - \frac{6z(z+D)^2}{R_2^5}\right\}, \tag{35}$$

where

$$R_1 = \sqrt{x^2+y^2+(z-D)^2}, \qquad R_2 = \sqrt{x^2+y^2+(z+D)^2}. \tag{36}$$

Let us seek to obtain the relationship between the moment of the nucleus C and the increment of the pressure ΔP or the source volume ΔV. According to SASAI (1979), the normal stress σ_{RR} across the source wall is given by

$$\sigma_{RR} = \frac{2C}{a^3}, \tag{37}$$

which should be balanced with the internal hydrostatic pressure ΔP. We follow the standard sign convention for stress, i.e. compression is negative, and we have

$$C = -\frac{1}{2}a^3\Delta P. \tag{38}$$

Using either of eq. (33) to eq. (35), we can calculate the increment of the source radius Δa when an internal hydrostatic pressure ΔP is applied, which is given by the displacement at an arbitrary point on the source sphere, for example,

$$\Delta a = |u_z(0,0,D-a)| = \frac{C}{2\mu}\frac{1}{a^2} \quad \left(\frac{a}{D} << 1\right). \tag{39}$$

The incremental source volume is estimated as

$$\Delta V = 4\pi a^2 \Delta a = \frac{2\pi}{\mu} C. \tag{40}$$

Thus the moment C can be expressed in terms of ΔV or ΔP as

$$C = \frac{\mu}{2\pi}\Delta V = -\frac{a^3}{2}\Delta P. \tag{41}$$

Two kind of sources are assumed, i.e.
(1) The center of dilatation (COD):

$$C = \frac{\mu}{2\pi}\Delta V, \tag{42}$$

where the incremental source volume ΔV is kept constant, while
(2) The center of pressure (COP):

$$C = -\frac{a^3}{2}\Delta P, \tag{43}$$

where the incremental pressure ΔP remains constant.

7 Piezomagnetic potential in the elastic medium

The piezomagnetic field due to the Mogi model was obtained by SASAI (1991a), which was a corrected version of the point source solution by SASAI (1979). The piezomagnetic potential due to a point source of the Mogi model was obtained on the basis of the displacement field eq. (33) to (35) (SASAI, 1991a) as:

$$\begin{aligned}\frac{2\mu}{C}W_x = 4\pi C_x \Bigg[&\frac{\mu}{3\lambda+2\mu}\left(\frac{x_0}{\rho_0}-\frac{x_0}{\rho_3^3}\right)+\frac{6(\lambda+\mu)}{3\lambda+2\mu}H\frac{3x_0D_3}{\rho_3^5} \\ &+\begin{cases}\frac{\lambda+\mu}{3\lambda+2\mu}\left(\frac{x_0}{\rho_1^3}-\frac{3x_0}{\rho_2^3}\right)\Big] & (H>D) \\ 0 \quad \Big] & (H<D)\end{cases}\end{aligned} \tag{44}$$

$$\begin{aligned}\frac{2\mu}{C}W_z = 4\pi C_z \Bigg[&-\frac{\mu}{3\lambda+2\mu}\left(\frac{D_1}{\rho_1^3}-\frac{D_3}{\rho_3^3}\right)+\frac{6(\lambda+\mu)}{3\lambda+2\mu}H\left(-\frac{1}{\rho_3^3}+\frac{3D_3^2}{\rho_3^5}\right) \\ &+\begin{cases}-\frac{\lambda+\mu}{3\lambda+2\mu}\left(\frac{D_1}{\rho_1^3}+\frac{3D_2}{\rho_2^3}\right)\Big] & (H>D) \\ 0 \quad \Big] & (H<D)\end{cases}\end{aligned} \tag{45}$$

where

$$\rho_i = (x_0^2 + y_0^2 + D_i^2)^{1/2}, \tag{46}$$

and

$$D_1 = D - z_0,\ D_2 = 2H - D - z_0,\ D_3 = 2H + D - z_0. \tag{47}$$

Since C_k is given by

$$C_k = \frac{1}{2}\beta J_k \mu \frac{3\lambda+2\mu}{\lambda+\mu}, \tag{48}$$

eq.(44) and (45) are rewritten as follows:

$$
\begin{aligned}
W_x &= \pi\beta J_x C \left[\frac{\mu}{\lambda+\mu}\left(\frac{x_0}{\rho_0} - \frac{x_0}{\rho_3^3}\right) + 18H\frac{x_0 D_3}{\rho_3^5}\right. \\
&\quad + \left\{ \begin{array}{ll} \left.\left(\frac{x_0}{\rho_1^3} - \frac{3x_0}{\rho_2^3}\right)\right] & (H > D) \\ \quad 0 \quad] & (H < D) \end{array} \right.
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
W_z &= \pi\beta J_z C \left[-\frac{\mu}{\lambda+\mu}\left(\frac{D_1}{\rho_1^3} - \frac{D_3}{\rho_3^3}\right) + 6H\left(-\frac{1}{\rho_3^3} + \frac{3D_3^2}{\rho_3^5}\right)\right. \\
&\quad + \left\{ \begin{array}{ll} \left.-\left(\frac{D_1}{\rho_1^3} + \frac{3D_2}{\rho_2^3}\right)\right] & (H > D) \\ \quad 0 \quad] & (H < D) \end{array} \right.
\end{aligned}
\tag{50}
$$

Depending on the type of the source, we may choose (42) for COD or (43) for COP as the moment C.

8 Application of the correspondence principle

We are to find the time-dependent piezomagnetic potential for the viscoelastic medium. We follow BONAFEDE *et al.* (1986), in which they adopted a standard linear solid (SLS) rheology for the shear modulus μ, while the bulk modulus K was left as a constant. We assume that an abrupt increase in the source volume ΔV or the internal pressure ΔP takes place in the source sphere, which is represented by a Heaviside function $g(t)$ in eq. (23). In the piezomagnetic potential eq. (49) and (50), only the elastic constants λ and μ as well as C in the case of COD, i.e. eq. (42), show the viscoelastic behavior. We can apply the correspondence principle to these equations and obtain the time-dependent behavior of the piezomagnetic field.

In the case of COP, C contains no elastic constants. Also we find in eq. (49) and (50) that there exist terms without elastic constants. This implies that a portion of the piezomagnetic field is time-independent and that it will never vanish forever. Physical interpretation for this is that an incremental pressure ΔP continuously applies to the source surface to always generate shear stress within the viscoelastic medium. Hence the source sphere continues to expand and will finally break. The COP model is somewhat physically unrealistic.

On the otherhand, in the case of COD, all the terms are multiplied by μ, and the piezomagnetic field will more or less diminish owing to the shear stress relaxation. In particular for Maxwell rheology, the shear stress and hence the piezomagnetic field will completely disappear. The wall of the source sphere stops after the initial expansion, which is likely to occur in the actual Earth.

We apply the correspondence principle to eq. (49) and (50) to obtain their Laplace transform. Looking for elastic constants in eq. (49) and (50) together with (42) and (43), we find only three coefficients which contain λ and μ, i.e. $\mu^2/(\lambda+\mu)$ and μ for COD and $\mu/(\lambda+\mu)$ for COP. Taking into account that $K = \lambda + \frac{2}{3}\mu$, we denote these coefficients as follows:

$$\alpha_1 = \frac{\mu}{\lambda+\mu} = \frac{3\mu}{3K+\mu}, \tag{51}$$

$$\alpha_2 = \mu\alpha_1 = \frac{3\mu^2}{3K+\mu}. \tag{52}$$

With the aid of eq. (26), all the necessary Laplace transforms are given as follows:

$$\tilde{g}(s) = \frac{1}{s} \tag{53}$$

$$\frac{1}{s}\tilde{\mu}(s) = \mu_1 \left[(1 - \frac{\mu_1}{\eta A})\frac{1}{s} + \frac{\mu_1}{\eta A}\frac{1}{s+A} \right] \tag{54}$$

$$\frac{1}{s}\tilde{\alpha}_1(s) = \frac{3\mu_1}{3K+\mu_1}\left[\{1 - \frac{3K\mu_1}{(3K+\mu_1)\eta B}\}\frac{1}{s} + \frac{3K\mu_1}{(3K+\mu_1)\eta B}\frac{1}{s+B} \right] \tag{55}$$

$$\begin{aligned}\frac{1}{s}\tilde{\alpha}_2(s) = \frac{3\mu_1^2}{3K+\mu_1}\Bigg[&\{1 - \frac{3K+\mu_1}{\mu_1+\mu_2} + \frac{9K^2}{3K(\mu_1+\mu_2)+\mu_1\mu_2}\}\frac{1}{s} \\ &+ \frac{3K+\mu_1}{\mu_1+\mu_2}\frac{1}{s+A} - \frac{9K^2}{3K(\mu_1+\mu_2)+\mu_1\mu_2}\frac{1}{s+B} \Bigg],\end{aligned} \tag{56}$$

where

$$A = \frac{\mu_1+\mu_2}{\eta} \tag{57}$$

$$B = \frac{3K(\mu_1+\mu_2)+\mu_1\mu_2}{(3K+\mu_1)\eta}. \tag{58}$$

Note that BONAFEDE *et al.*'s (1986) time unit τ is defined by B^{-1}, which is the typical relaxation time of the COP model.

The inverse Laplace transforms of these functions can be easily obtained since we have

$$\mathcal{L}^{-1}\left(\frac{1}{s+p}\right) = e^{-pt} \tag{59}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = g(t). \tag{60}$$

We find that the time variation of the piezomagnetic field shows: (1) the exponential decay with time constants A^{-1} and B^{-1} plus the time-invariant component for COD source, (2) the exponential decay with a time constant B^{-1} plus the time-invariant component for COP, and (3) the initial value at $t = 0$ completely identical to the elastic solution. In particular, for a Maxwell solid, both the coefficients of $1/s$ in eq. (54) and (56) become zero by putting $\mu_2 = 0$. Then we find that: (4) for a Maxwell solid, the solution for COD source in the time domain lacks the term containing $g(t)$, which implies that the magnetic field diminishes to zero when $t \to \infty$, while (5) even for a Maxwell solid, the solution for COP source remains finite when $t \to \infty$ because it contains some terms multiplied by $g(t)$.

9 Discussion

The present method can be easily applied to some tectonomagnetic models in an elastic half-space, i.e. the Mogi model with a finite spherical source (SASAI, 1991a), vertical rectangular strike-slip and tensile faults (SASAI, 1991b), inclined shear and tensile faults (UTSUGI *et al.*, 2000) and a uniform circular load or the dam-magnetic effect (SASAI, 1986). However, a viscoelastic half-space is a too crude approximation to the actual earth. More realistic would be a layered earth with the mixture of elastic and viscoelastic ones. OKUBO and OSHIMAN

(2004) presented the piezomagnetic field due to the Mogi model in a layered elastic earth. They derived the solution by a numerical volume integral, to which the present method is not applicable. For a point source or strain nuclei of elastic dislocation problems, we presume, we can obtain analytic formulas in a layered earth with the propagator matrix technique which was employed by OKUBO and OSHIMAN (2004). Then the present method will be available to investigate the piezomagnetic field in a viscoelastic earth. This may be a rather laborious work, but should make a breakthrough to further tectonomagnetic studies.

Another interesting problem is to compare the time variation of the piezomagnetic field with that of the gravity in the viscoelastic medium. This is because the gravity due to density changes is caused solely by the hydrostatic pressure, which is complementary to the piezomagnetic change produced by the deviatoric stress alone. Combined observations of the gravity and the magnetic field by recent high-accuracy absolute measurement systems can be a useful tool to investigate the viscoelastic behavior of the earth.

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